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TWO-DIMENSIONAL, LIFTING WINGS OF MINIMUM DRAG IN HYPERSONIC FLOW[*]

by

DAVID G. HULL[**]

SUMMARY

The problem of minimizing the drag of a slender, flat-top, two-dimensional, lifting wing in hypersonic flow is considered under the assumptions that the pressure coefficient is Newtonian and the skin-friction coefficient is constant. The indirect methods of the calculus of variations are employed, and the necessary conditions to be satisfied by an optimum airfoil are derived for conditions imposed on the lift, the pitching moment, the profile area, the chord length, and the thickness. Then, the following particular cases are analyzed: (a) given lift, (b) given lift and chord length,

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[**] Research Associate in Astronautics, Department of Mechanical and Aerospace Engineering and Materials Science, Rice University, Houston, Texas.

(c) given lift and thickness, (d) given lift and profile area, and (e) given lift, pitching moment, and chord length. In all of these cases, analytical expressions are presented for the geometry of the optimum airfoil and the aerodynamic drag.

1. INTRODUCTION

In Refs. 1 through 5, the minimization of the drag of a two-dimensional wing in Newtonian flow was considered under the assumptions that the airfoil is symmetric and the angle of attack is zero, so that the lift is zero. Since the operational qualities of a hypersonic vehicle depend to a large extent on its lifting characteristics, the more important problem of minimizing the drag of a lifting wing is investigated here.

One way to approach this problem is to place a symmetric airfoil at an angle of attack. Another way is to study airfoils such that the upper and lower surfaces are asymmetric. The latter approach is used here, with the further assumption that every element of the upper surface is parallel to the undisturbed flow direction and every element of the lower surface "sees" the flow. For these flat-top wings, the variational problem consists of determining the lower surface so as to minimize the drag for various conditions imposed on the lift, the pitching moment, the profile area, the chord length, and the thickness. The complete list of hypotheses is as follows: (a) the wing is two-dimensional; (b) the upper surface is flat and parallel to the undisturbed flow direction; (c) the wing is slender in the chordwise sense; (d) the pressure coefficient is Newtonian; (e) the skin-friction coefficient is constant; and (f) the effect of the tangential forces on the lift and the pitching moment is neglected.

2. AERODYNAMIC AND GEOMETRIC QUANTITIES

In order to relate the aerodynamic and geometric quantities of a two-dimensional, flat-top wing to its geometry, the following Cartesian coordinate system Oxz is introduced: the origin O is the leading edge; the x-axis is in the direction of the undisturbed flow; and the z-axis is normal to the x-axis and positive downward (Fig. 1).

If hypotheses (a) through (f) are considered and if the lower surface is represented by the relationship $z = z(x)$, the drag per unit span D, the lift per unit span L, the pitching moment per unit span M, and the profile area A are given by (Ref. 6)

$$
\begin{aligned}
D/2q &= \int_{x_i}^{x_f} \left(\dot{z}^3 + C_f\right) dx \\
L/2q &= \int_{x_i}^{x_f} \dot{z}^2 \, dx \\
M/2q &= \int_{x_i}^{x_f} x\,\dot{z}^2 \, dx \\
A &= \int_{x_i}^{x_f} z \, dx
\end{aligned}
\qquad (1)
$$

where q is the free-stream dynamic pressure, $\dot{z}$ the derivative dz/dx, and C_f is the constant skin-friction coefficient. The end coordinates are represented by the relations

$$x_i = 0 \quad , \quad z_i = 0$$
$$x_f = c \quad , \quad z_f = t \tag{2}$$

where the chord length c and the trailing edge thickness t can either be arbitrarily prescribed or free.

Other aerodynamic and geometric quantities of interest are the lift-to-drag ratio $E = L/D$, the position of the center of pressure $x_o = M/L$, and the thickness ratio $\tau = t/c$.

3. MINIMAL PROBLEM

The problem of minimizing the drag for arbitrary values imposed on the lift, the pitching moment, the profile area, the chord length, and the thickness is now formulated as follows: "In the class of functions z(x) which satisfy the integral constraints (1-2) through (1-4) and the prescribed boundary conditions, find that particular function which minimizes the integral (1-1)." According to standard variational procedures (see, for instance, Chapter 1 of Ref. 7), this problem is equivalent to that of minimizing the functional

$$I = \int_{x_i}^{x_f} F(x, z, \dot{z}, \lambda_1, \lambda_2, \lambda_3)\, dx \tag{3}$$

subject to the constraints (1-2) through (1-4) and the prescribed boundary conditions with the understanding that the fundamental function F is defined as

$$F = \dot{z}^3 + C_f + \lambda_1 \dot{z}^2 + \lambda_2 x \dot{z}^2 + \lambda_3 z \tag{4}$$

where λ_1, λ_2, λ_3 denote constant Lagrange multipliers.

4. NECESSARY CONDITIONS

The function z(x) which extremizes the functional (3) must be a solution of the Euler equation

$$dF_{\dot{z}}/dx - F_z = 0 \tag{5}$$

Its explicit form

$$\frac{d}{dx}\,(3\dot{z}^2 + 2\lambda_1\dot{z} + 2\lambda_2 x\,\dot{z}) - \lambda_3 = 0 \tag{6}$$

admits the first integral

$$3\dot{z}^2 + 2\lambda_1\dot{z} + 2\lambda_2 x\,\dot{z} - \lambda_3 x = C \tag{7}$$

where C is a constant. While a second integral can be obtained, it is more convenient to derive it when analyzing particular cases.

The general solution of the Euler equation involves two integration constants whose values are determined by applying the prescribed boundary conditions and the natural boundary conditions. The latter are obtained from the transversality condition

$$\Big[(F - \dot{z}F_{\dot{z}})\,\delta x + F_{\dot{z}}\,\delta z\Big]_i^f = 0 \tag{8}$$

which must be satisfied for every system of variations consistent with the prescribed boundary conditions. The explicit form of Eq. (8) is the following:

$$\left[(-2\dot{z}^3 + C_f - \lambda_1\dot{z}^2 - \lambda_2 x\dot{z}^2 + \lambda_3 z)\,\delta x + (3\dot{z}^2 + 2\lambda_1\dot{z} + 2\lambda_2 x\dot{z})\,\delta z\right]_i^f = 0 \tag{9}$$

where $\delta x_i = \delta z_i = 0$, owing to Eqs. (2-1). As a consequence, the natural boundary conditions are given by

$$(-2\dot{z}^3 + C_f - \lambda_1\dot{z}^2 - \lambda_2 x\dot{z}^2 + \lambda_3 z)_f = 0 \tag{10}$$

if the chord length is free and

$$(3\dot{z}^2 + 2\lambda_1\dot{z} + 2\lambda_2 x\dot{z})_f = 0 \tag{11}$$

if the thickness is free.

Once the solution of the Euler equation is obtained, it is necessary to verify that it actually minimizes the functional (3). In this connection, the Legendre necessary condition

$$F_{\dot{z}\dot{z}} \geq 0 \tag{12}$$

must be satisfied and is equivalent to

$$3\dot{z} + \lambda_1 + \lambda_2 x \geq 0 \tag{13}$$

5. NONDIMENSIONAL VARIABLES

In the following sections, several particular cases are analyzed with the aid of the previous necessary conditions. Since the main purpose of a wing is to generate enough lift to maintain a vehicle in flight, the lift is prescribed in every case. In order to present the results in the most compact way, it is convenient to introduce the nondimensional coordinates

$$\xi = x/c \quad , \quad \zeta = z/t \tag{14}$$

and the nondimensional variables

$$\begin{aligned} E_* &= E\,C_f^{1/3} \\ \xi_o &= x_o/c \\ c_* &= c(q/L)\;C_f^{2/3} \\ t_* &= t(q/L)\;C_f^{1/3} \\ A_* &= A(q/L)^2\;C_f \\ \tau_* &= \tau\,C_f^{-1/3} \end{aligned} \tag{15}$$

Incidentally, the quantity E_* is inversely proportional to the drag and, for that matter, is a maximum when the drag is a minimum.

6. GIVEN LIFT

If the lift is prescribed while the pitching moment and the profile area are free ($\lambda_2 = \lambda_3 = 0$), the first integral (7) reduces to

$$3\dot{z}^2 + 2\lambda_1 \dot{z} = C \tag{16}$$

and implies that

$$\dot{z} = \text{Const} = \tau \tag{17}$$

Hence, the optimum two-dimensional wing is the wedge

$$\zeta = \xi \tag{18}$$

The evaluation of the drag integral (1-1) and the lift integral (1-2) yields the relationships

$$D = 2qc(\tau^3 + C_f) \quad , \quad L = 2qc\tau^2 \tag{19}$$

which imply that

$$E_* = \tau_*^2/(\tau_*^3 + 1) \tag{20}$$

Since the chord and the thickness are free, the natural boundary conditions (10) and (11) hold and, because of Eq. (17), can be rewritten as

$$2\tau^3 - C_f + \lambda_1 \tau^2 = 0 \quad , \quad 3\tau^2 + 2\lambda_1 \tau = 0 \tag{21}$$

Upon eliminating the multiplier λ_1 from these equations, we see that the optimum thickness ratio is given by

$$\tau_* = \sqrt[3]{2} \cong 1.26 \tag{22}$$

and is such that the friction drag is one-third of the total drag. The associated thickness and chord are given by

$$t_* = 1/\sqrt[3]{16} \cong 0.397 \quad , \quad c_* = 1/\sqrt[3]{32} \cong 0.315 \tag{23}$$

Finally, the lift-to-drag ratio (20) becomes

$$E_* = \sqrt[3]{4}\,/3 \cong 0.529 \tag{24}$$

Equation (24) represents the highest possible value of the lift-to-drag ratio which can be obtained with a flat-top wing (Refs. 6 and 8). Should the wing be required to satisfy a certain number of geometric and/or aerodynamic constraints, a decrease in the lift-to-drag ratio would occur with respect to that predicted by Eq. (24).

7. GIVEN LIFT AND CHORD

If the lift and the chord length are prescribed while the pitching moment and the profile area are free, Eqs. (16) through (20) are still valid. The optimum thickness ratio is obtained from the relationship (19-2) as follows:

$$\tau_* = (1/2c_*)^{1/2} \tag{25}$$

Thus, the lift-to-drag ratio (20) becomes (Fig. 2)

$$E_* = \frac{(2c_*)^{1/2}}{(2c_*)^{3/2} + 1} \tag{26}$$

and achieves the maximum value (24) for the value of c_* defined by Eq. (23-2).

8. GIVEN LIFT AND THICKNESS

If the lift and the thickness are prescribed while the pitching moment and the profile area are free, Eqs. (16) through (20) are still valid. The optimum thickness ratio is obtained from the relationship (19-2) as follows:

$$\tau_* = 1/2t_* \tag{27}$$

Thus, the lift-to-drag ratio (20) becomes (Fig. 3)

$$E_* = \frac{2t_*}{(2t_*)^3 + 1} \tag{28}$$

and achieves the maximum value (24) for the value of t_* defined by Eq. (23-1).

9. GIVEN LIFT AND PROFILE AREA

If the lift and the profile area are prescribed while the pitching moment is free ($\lambda_2 = 0$), the first integral (7) reduces to

$$3\dot{z}^2 + 2\lambda_1 \dot{z} - \lambda_3 x = C \tag{29}$$

which, when applied at the trailing edge and combined with the natural boundary condition (11), supplies the relation

$$C = -\lambda_3 c \tag{30}$$

At the same time, the natural boundary condition (11) implies that two classes of solutions exist

$$\begin{array}{ll} \underline{\text{Class I}} & \dot{z}_f = -2\lambda_1/3 \\ \underline{\text{Class II}} & \dot{z}_f = 0 \end{array} \tag{31}$$

For the former, the Legendre condition requires that $\lambda_1 \leq 0$, while for the latter, it requires that $\lambda_1 \geq 0$.

The quadratic equation (29) is solved by

$$\dot{z} = (1/3)\left\{-\lambda_1 + \left[\lambda_1^2 - 3\lambda_3 (c - x)\right]^{1/2}\right\} \tag{32}$$

where the plus sign is chosen so that the Legendre condition is satisfied. Integrating this differential equation subject to the initial conditions (2-1), one obtains the relation

$$z = -(\lambda_1/3)x - (2/27\lambda_3)\left\{\left[\lambda_1^2 - 3\lambda_3 c\right]^{3/2} - \left[\lambda_1^2 - 3\lambda_3(c - x)\right]^{3/2}\right\} \tag{33}$$

which, applied at the trailing edge, implies that the thickness is given by

$$t = -(\lambda_1/3)c - (2/27\lambda_3)\left\{\left[\lambda_1^2 - 3\lambda_3 c\right]^{3/2} \pm \lambda_1^3\right\} \tag{34}$$

where the upper sign is valid for solutions of Class I and the lower sign, for solutions of Class II. At this point, we define the quantity

$$\alpha = 3\lambda_3 c/\lambda_1^2 \tag{35}$$

and the function

$$G(\xi,\alpha) = 3\alpha\xi \mp 2\left\{[1 - \alpha]^{3/2} - [1 - \alpha(1 - \xi)]^{3/2}\right\} \tag{36}$$

so that Eqs. (33) and (34) can be rewritten as

$$\begin{aligned} z &= -(\lambda_1^3/27\lambda_3)\ G(\xi,\alpha) \\ t &= -(\lambda_1^3/27\lambda_3)\ G(1,\alpha) \end{aligned} \tag{37}$$

and imply that

$$\zeta = G(\xi, \alpha)/G(1, \alpha) \tag{38}$$

The next step is to relate the quantity α to the prescribed values of the lift and the profile area. In this connection, the combined use of the natural boundary condition (10), Eqs. (31), and Eq. (37-2) leads to the relationship

$$\tau_* = [G(1,\alpha)/3\alpha][4(2 - n) - G(1,\alpha)]^{-1/3} \tag{39}$$

where n = 1 for solutions of Class I and n = 2 for solutions of Class II. Furthermore, substituting the shape equation (38) into the lift constraint (1-2) and the profile area constraint (1-4) and accounting for Eq. (39), one obtains the relations

$$c_* = \frac{3\alpha[4(2 - n) - G(1,\alpha)]^{2/3}}{12\alpha - 3\alpha^2 \mp 8[(1 - \alpha)^{3/2} - 1]}$$

$$A_* = \frac{3}{10} \frac{15\alpha^2 \mp 4[(2 + 3\alpha)(1 - \alpha)^{3/2} - 2]}{\left\{12\alpha - 3\alpha^2 \mp 8[(1 - \alpha)^{3/2} - 1]\right\}^2}[4(2-n) - G(1,\alpha)] \tag{40}$$

where

$$\underline{\text{Class I}} \qquad 0 \le A_* \le 4/5 \quad , \quad 3/4 \ge \alpha \ge -\infty$$

$$\underline{\text{Class II}} \qquad 4/5 \le A_* \le \infty \quad , \quad -\infty \le \alpha \le 0 \tag{41}$$

The associated lift-to-drag ratio is given by

$$E_* = \frac{5}{9} \frac{[4(2 - n) - G(1, \alpha)]^{1/3} \{12\alpha - 3\alpha^2 \mp 8[(1 - \alpha)^{3/2} - 1]\}}{20\alpha - 15\alpha^2 \mp 8[(1 - \alpha)^{5/2} - 1]} \tag{42}$$

The final step consists of eliminating the quantity α from Eqs. (38) through (42). If this is done, one obtains the parametric functional relationships

$$\zeta = f_1(\xi, A_*) \tag{43}$$

and

$$c_* = f_2(A_*) \quad , \quad \tau_* = f_3(A_*) \quad , \quad E_* = f_4(A_*) \tag{44}$$

which are plotted in Figs. 4 through 7. Incidentally, the lift-to-drag ratio (44-3) achieves the maximum value (24) for $A_* = 1/16$.

10. GIVEN LIFT, PITCHING MOMENT, AND CHORD

If the lift and the pitching moment are prescribed while the profile area is free ($\lambda_3 = 0$), the first integral (7) reduces to

$$\dot{z}(3\dot{z} + 2\lambda_1 + 2\lambda_2 x) = C \tag{45}$$

with $C = 0$, owing to the natural boundary condition (11). The integration of this equation subject to the initial conditions (2-1) yields the relationship

$$z = -(2\lambda_1/3)x - (\lambda_2/3)x^2 \tag{46}$$

which, at the trailing edge, becomes

$$t = -(2\lambda_1/3)c - (\lambda_2/3)c^2 \tag{47}$$

In these relations, the Legendre condition requires that $\lambda_1 \leq 0$. At this point, we define the quantity

$$\beta = \lambda_2 c/\lambda_1 \tag{48}$$

and rewrite the previous equations as

$$\begin{aligned} z &= -(c\lambda_1/3)(2\xi + \beta\xi^2) \\ t &= -(c\lambda_1/3)(2 + \beta) \end{aligned} \tag{49}$$

with the implication that

$$\zeta = (2\xi + \beta\xi^2)/(2 + \beta) \tag{50}$$

The next step is to relate the quantity β to the prescribed values of the lift and the pitching moment. In this connection, if the shape equation (50) is substituted into the lift constraint (1-2) and the moment constraint (1-3), one obtains the relations

$$\begin{aligned} \tau_* \sqrt{c_*} &= (\beta + 2)[3/8\,(\beta^2 + 3\beta + 3)]^{1/2} \\ \xi_o &= (3\beta^2 + 8\beta + 6)/4(\beta^2 + 3\beta + 3) \end{aligned} \tag{51}$$

where

$$1/4 \le \xi_o \le 3/4 \quad , \quad -1 \le \beta \le \infty \tag{52}$$

The associated lift-to-drag ratio is given by

$$E_* = \frac{(2c_*)^{1/2}}{(2c_*)^{3/2} + H(\beta)} \tag{53}$$

where

$$H(\beta) = (3/4)\sqrt{3}\,(\beta^3 + 4\beta^2 + 6\beta + 4)(\beta^2 + 3\beta + 3)^{-3/2} \tag{54}$$

The final step consists of eliminating the quantity β from Eqs. (50) through (53). If this is done, one obtains the parametric functional relationships

$$\zeta = g_1(\xi, \xi_o) \tag{55}$$

and

$$\tau_* \sqrt{c_*} = g_2(\xi_o) \quad , \quad E_* = g_3(\xi_o, c_*) \tag{56}$$

which are plotted in Figs. 8 through 10. Incidentally, the lift-to-drag ratio (56) achieves the absolute maximum value (24) when

$$\xi_o = 1/2 \quad , \quad c_* = 2^{-5/3} \tag{57}$$

In closing, it is emphasized that the present solutions are valid only if the prescribed center of pressure lies between 25% and 75% of the chord length. The remaining solutions can be obtained by combining the regular shape (50) with zero-slope shapes; however, these solutions are not derived here since the range of ξ_o covered in Figs. 8 through 10 is sufficient for most practical applications.

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LIST OF CAPTIONS

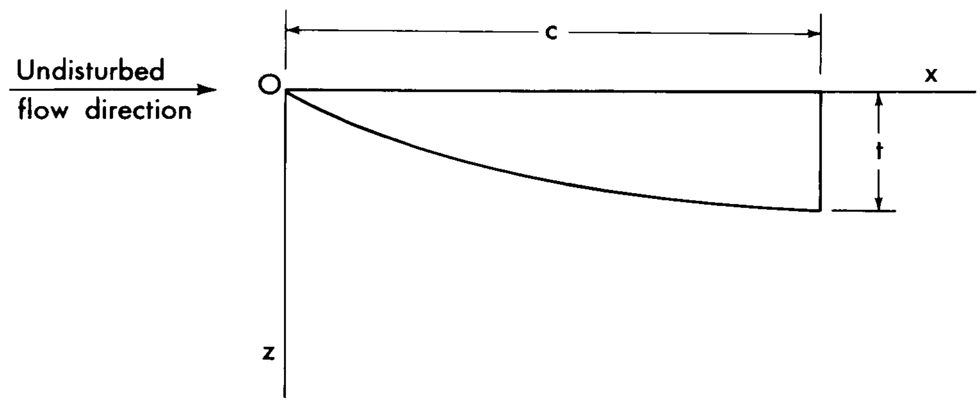

Fig. 1. Coordinate system.

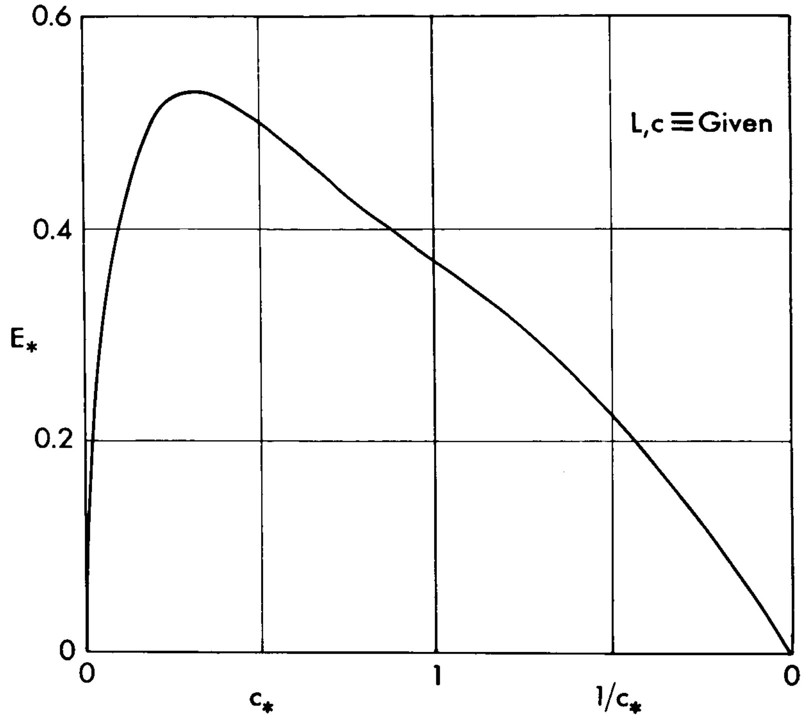

Fig. 2. Maximum lift-to-drag ratio.

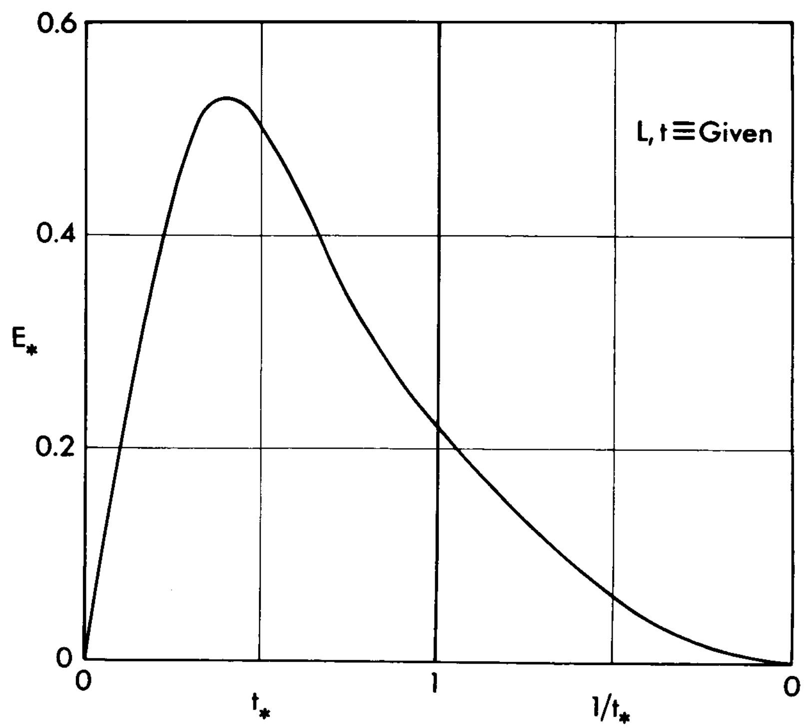

Fig. 3. Maximum lift-to-drag ratio.

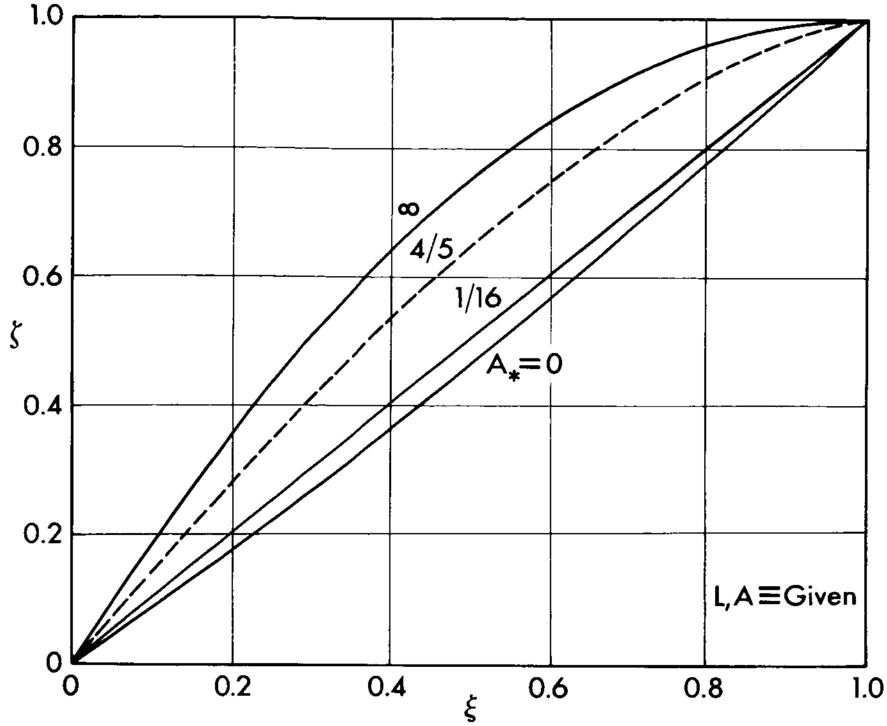

Fig. 4. Optimum shape.

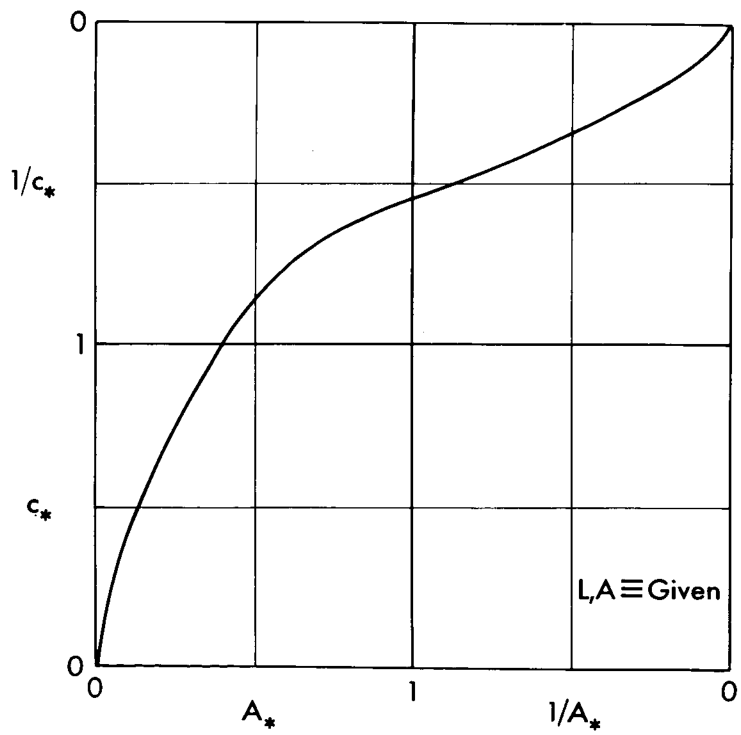

Fig. 5. Optimum chord length.

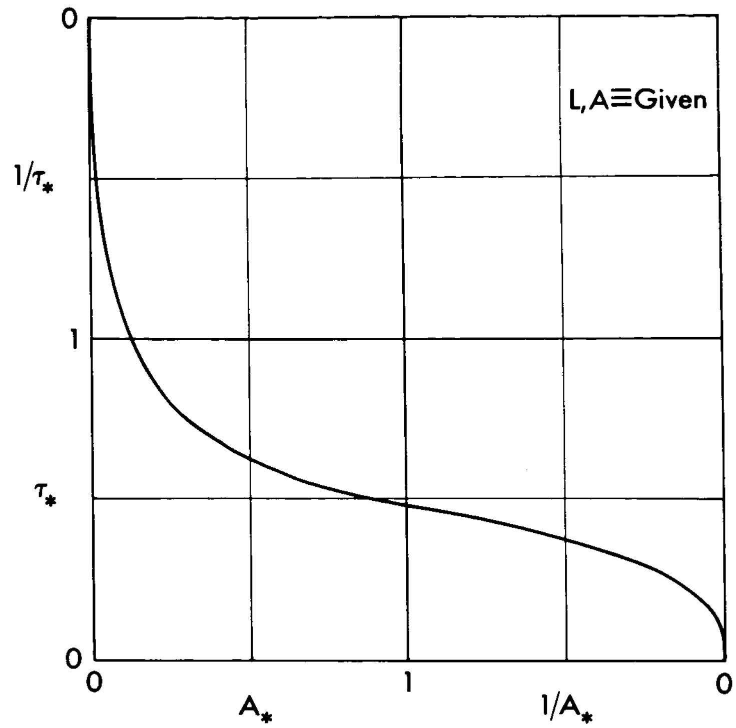

Fig. 6. Optimum thickness ratio.

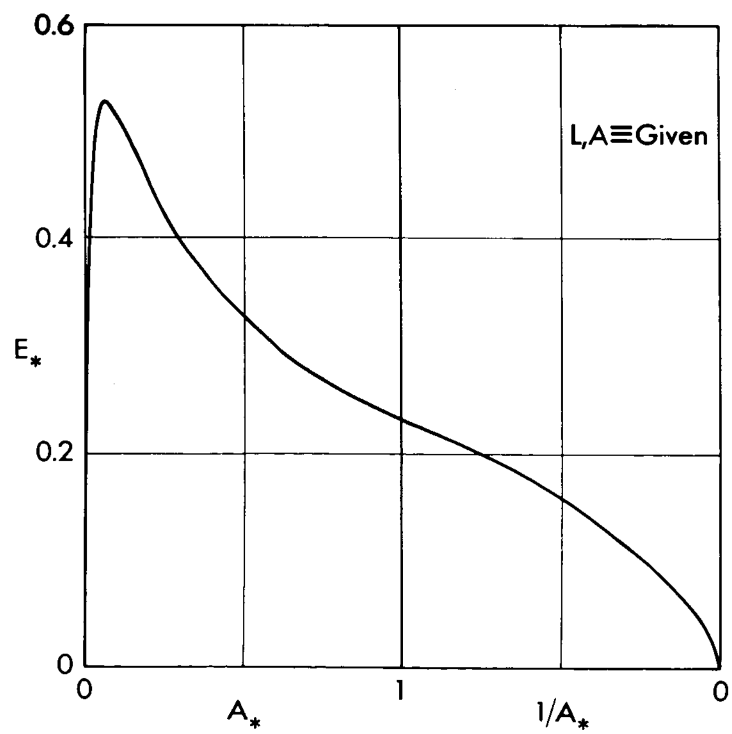

Fig. 7. Maximum lift-to-drag ratio.

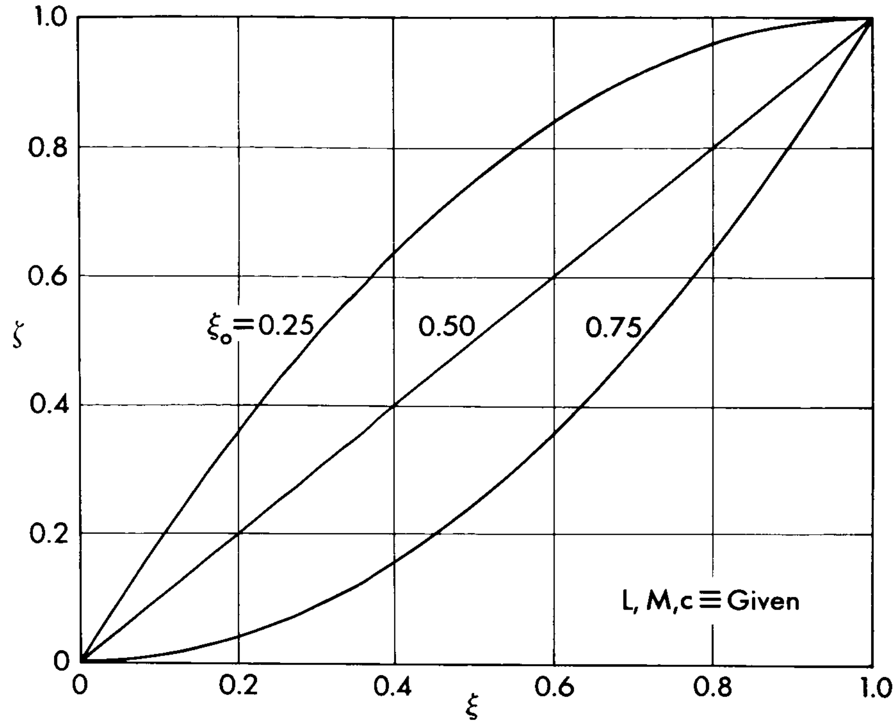

Fig. 8. Optimum shape.

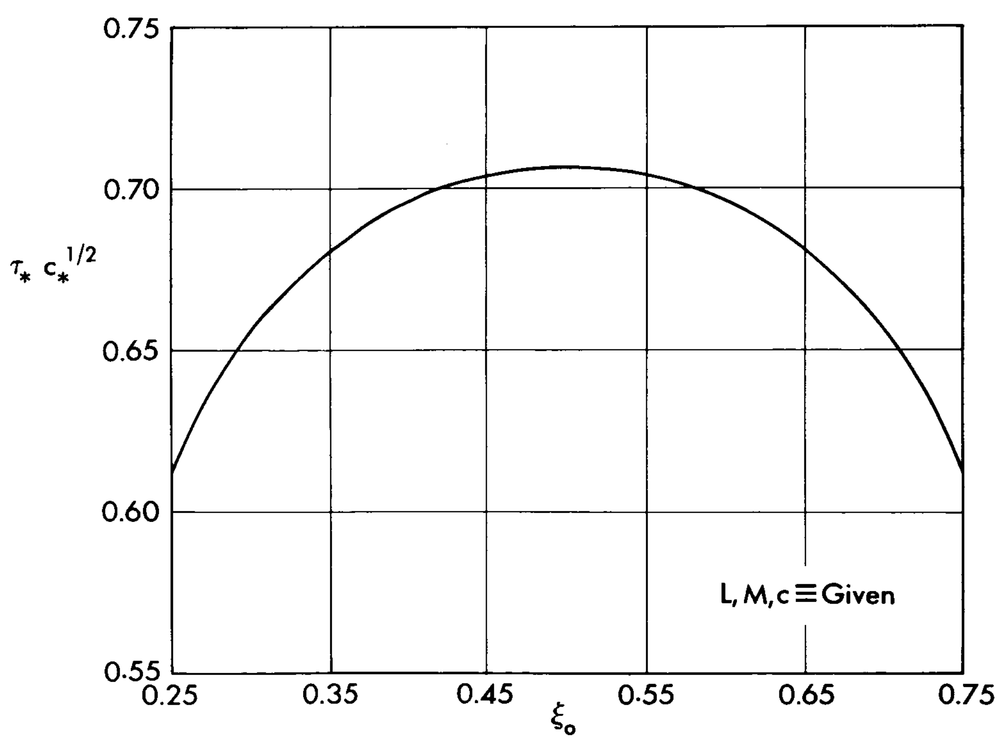

Fig. 9. Optimum thickness ratio.

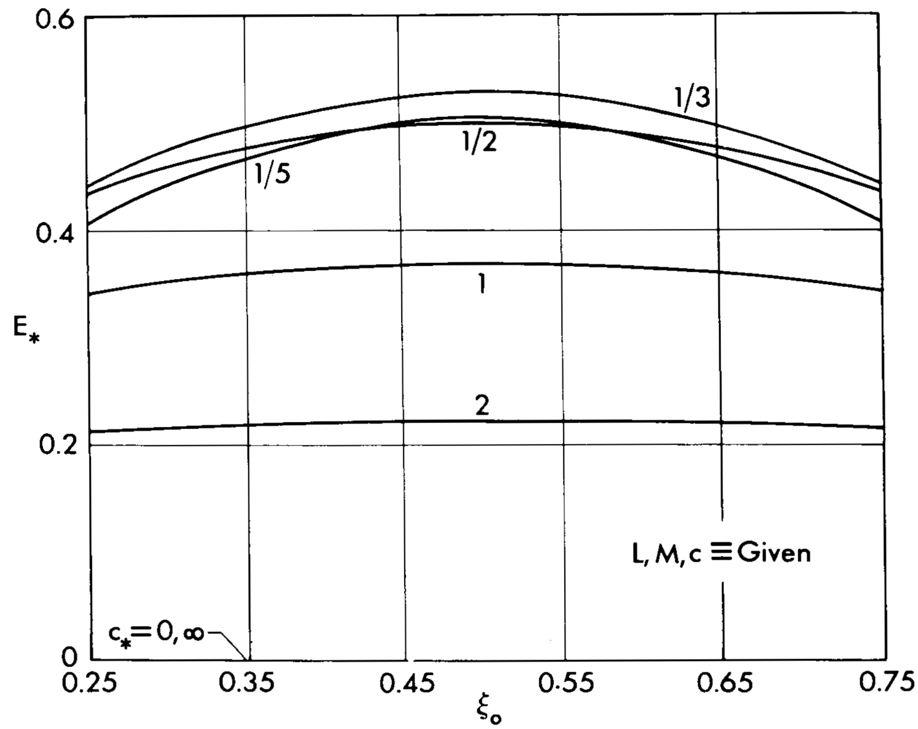

Fig. 10. Maximum lift-to-drag ratio.